RESPONSE OF A STRATIFIED VISCOELASTIC HALF-SPACE UNDER A SURFACE PERIODIC LOAD

Summary. The local crustal stresses and deformations induced by near-field loading are studied. A stratified half-space is considered, being more convenient from the computational point of view than the spherical half-space when treating problems such as the near-sea tidal loading effects in earth tide observations.

The equations of motion with proper boundary conditions are derived for the viscoelastic case. Vertical and horizontal surface displacements are given as functions of the wavenumber for two kinds of earth rheological models (one is a function of the viscosity coefficient μ, the other is a function of the quality factor Q) and for one diurnal (O1) and one semidiurnal (M2) wave. Although the rheological models are very different, the amplitudes of the effects are very close to the corresponding pure elastic cases. An evident phase-lag of about 0.4° occurs at about 80 km from the load for the model PREM, mostly generated by a low Q asthenosphere in the upper mantle.

Received November 11, 1986

1. Introduction

The study of the inner rheological and geometrical structure of the earth obviously involves the treatment of inverse problems. The equations of motion are used widely in the solution of geophysical problems and this approach is also adopted in the present paper.

Several researchers have treated the problem of the response of the earth to an impulsive or periodic surface load, from about the beginning of this century. Lamb (1917) was the first to apply Green's function to this problem, that function having already been given by Boussinesq (1885) for a non-gravitating elastic homogeneous half-space loaded by a point force in the static case. After Lamb, Nishimura (1932) attempted a stratification of the medium in order to obtain a better interpretation of the vertical deflections induced by a surface load. The Boussinesq solution has been extensively used in the past and is still valid as a first approximation in some geophysical cases involving a simple local structure of completely unknown composition.

In recent years, there has been a substantial increase in the memory and speed of electronic computers. This development has encouraged researchers to find theoretical solutions for more and more sophisticated models using numerical techniques. Two partially overlapping branches of research have been developed, both of them regard the stratification of the model from the rheological as well as from the geometrical point of view, and in the gravitating or in the non-gravitating case. The first considers a spherical earth model, the second a flat model. In the former global effects are studied and detected, whereas the latter is best used in examination of the local effects (Zadro, 1972; Zschau, 1976).

* On leave from the Institute of Geodesy and Geophysics of the Chinese Academy of Sciences
** Institute of Geodesy and Geophysics, Trieste University, Italy
In the former approach, Slichter and Caputo (1960) and Caputo (1961, 1962) made significant developments, followed by Longman (1962, 1963) and Farrell (1972) who used the theoretical formulae related to free oscillations (Alterman et al., 1959) in order to reconstruct the loading response signal and Love numbers for the elastic case. Zschau (1977), Li and Hsu (1985) later treated the same problem for the viscoelastic case. The present research follows the second approach mentioned above. Following the first attempts of Lamb and Nishimura, it is interesting to cite the work of Kuo (1969), also for its particular relevance to our work. He used the Thompson-Haskell matrix method in order to find the response of a layered nongravitating elastic half-space to a surface load. Although the theoretical method is straightforward and highly efficient, it has not had much success due to computational problems in the numerical approximation. Subsequently Peltier (1974) and Cathles (1975) studied the isostatic rebound problem by utilizing viscoelastic earth models, which are also of relevance to the present work.

Here, the equations of motion for a stratified viscoelastic and constant gravitating half-space are solved by a simple numerical method. Amplitude and phase Green's functions are given. The relations between the solutions and the earth model parameters, different wave numbers and frequency of the applied force are then discussed.

2. The transformed elastic equations of motion

We begin by discussing the elastic problem since the viscoelastic solutions corresponding to this situation can be obtained by direct application of the correspondence principle (Lee, 1955). To study the response to a near-field load, it is assumed that the earth is a stratified half-space; the elastic forces dominate the gravitational forces and the gravity can be considered constant. In this case, the linearized equation of motion in a cylindrical coordinate $r, \theta, z$ is

$$\nabla \cdot \tau + \varrho_{0} g \nabla \cdot s \hat{e}_{z} - \varrho_{0} g \nabla \cdot \mathbf{u} = -\frac{\partial^{2} s}{\partial t^{2}} \quad (1)$$

where $\varrho_{0}$ is the unperturbed density, $g$ the acceleration due to gravity and $s$ the deformation displacement vector

$$s = u (z, r) \hat{e}_{z} + v (z, r) \hat{e}_{r} + w (z, r) \hat{e}_{\theta}. \quad (2)$$

The cylindrical coordinate axis $\hat{e}_{z}$ is directed vertically upward, $\hat{e}_{r}$ refers to the cylindrical radius and $\hat{e}_{\theta}$ the azimuthal angle. The stress tensor $\tau$ due to perturbation obeys the isotropic elastic constitutive equation

$$\tau = \lambda (\nabla \cdot s) I + \mu \left( \nabla s + (\nabla s)^{\top} \right). \quad (3)$$

$\lambda$ and $\mu$ are the Lamé elastic parameters and $I$ is a unit tensor. For a stratified half-space, $\lambda (z), \mu (z)$, and $\varrho (z)$ are functions of depth $z$. To remove the time dependence from (1) a Fourier transform with respect to time is made. No confusion should arise when omitting the symbols of the Fourier transform, so the transformed equation of motion is

$$\nabla \cdot \tau + \varrho_{0} g \nabla \cdot s \hat{e}_{z} - \varrho_{0} g \nabla \cdot \mathbf{u} = -\omega^{2} \varrho_{0} s. \quad (4)$$

where $\omega$ is the angular frequency. The transformed form of (3) remains unchanged. Assuming the free surface to be stress-free except for a point force at the origin, and proceeding as in the spherical case, we introduce the cylindrical surface harmonic $J_m (kr) e^{im\theta}$. Here, $k$ is the wave number in $1/r$ units. Since the point load is axially symmetric, the solutions do not depend on $\theta$. The harmonic $J_m (kr) e^{im\theta}$ is reduced to the
zeroth order Bessel function \(J_0 (kr)\). Since \(-J_0' (x) = J_1 (x)\), the displacement solutions can be expressed as Hankel transforms of order zero and one with respect to the \(r\) variable. Let \(U, \ V\) be the components of the transformed displacement

\[
\begin{align*}
    u (z, r) &= \int_0^\infty U (z, k) J_0 (kr) \ k dk \\
    v (z, r) &= \int_0^\infty V (z, k) J_1 (kr) \ k dk .
\end{align*}
\]

By using the Hankel transforms also for the stress components of equation (3), from equation (4) the following two second order differential equations are obtained:

\[
\begin{align*}
    - \sigma \ \kappa k^2 - \lambda \ k U + \frac{\ddot{T}_{rr}}{T_{rr}} + \psi g k U + \omega^2 \psi \ V &= 0 \\
    - \mu \ U k^2 + \mu \ k V + \frac{\ddot{T}_{zz}}{T_{zz}} + \psi g k V + \omega^2 \psi \ U &= 0
\end{align*}
\]

where \(\sigma = \lambda + 2 \mu; \ T_{rr} \) and \(T_{zz}\) are respectively Hankel transforms of order zero and one of the components \(T_{rr}\) and \(T_{zz}\) of the stress tensor. The dot denotes depth differential \(\partial/\partial z\), unless stated otherwise.

We introduce a set of new variables

\[
\begin{align*}
    y_1 &= U, \quad y_2 = \lambda \ X + 2 \mu \ \dot{U}, \quad y_3 = V, \quad y_4 = \mu \left( \dot{V} - \ddot{U} \right)
\end{align*}
\]

where \(X = \nu k + \dot{U}\) is the Hankel transform of \(\nabla \cdot s\) of order zero. (6) is reduced to four 1st order homogenous linear equations:

\[
\begin{align*}
    \dot{y}_1 &= 1/\sigma \ \nu \ y_2 - \lambda/\sigma \ y_3 \\
    \dot{y}_2 &= - \omega^2 \psi \ y_1 - \psi g k y_3 - \lambda \ y_4 \\
    \dot{y}_3 &= k y_1 + 1/\mu \ y_4 \\
    \dot{y}_4 &= - \psi g k y_1 + \lambda \ k/\sigma \ y_2 + (4 \mu \ (\lambda + \mu) \ k^2/\sigma - \omega^2 \psi) \ y_3
\end{align*}
\]

Simplifying the equation set (8) by making them non-dimensional we obtain:

\[
\begin{bmatrix}
    \frac{\partial}{\partial z} \left( \begin{array}{c}
        k \lambda^* y_1 \\
        y_2 \\
        k \lambda^* y_3 \\
        y_4
    \end{array} \right)
\end{bmatrix} = k \begin{bmatrix}
    0 & 1/\sigma & -1/\sigma & 0 & k \lambda^* y_1 \\
    -A \dot{\lambda} / k^2 & 0 & -B \dot{\lambda} \dot{\psi} / k & -1 & y_2 \\
    1 & 0 & 0 & 1/\mu & k \lambda^* y_3 \\
    -B \dot{\lambda} \dot{\psi} / k & \lambda / \sigma & 4 \tilde{\mu} (\lambda + \tilde{\mu}) / \sigma & -A \ddot{\lambda} / k^2 & 0 & y_4
\end{bmatrix}
\]

where \(\tilde{\mu} = \mu/\lambda^*, \ \lambda = \lambda/\lambda^*, \ \dot{\lambda} = \dot{\lambda}/\lambda^*, \ \lambda^* \) and \(\psi^* \) are the Lamé constants for the lowest layer; \(\dot{\psi} = g/g^*, \ g^* \) the gravity of the earth's surface and \(A = \omega^2 \dot{\psi} / \lambda^*, \ B = g^* g^* / \lambda^*\). The differential equation (9) is now suitable for integration by the Runge-Kutta method.

3. The boundary conditions and the initial vector

Consider a unit mass distributed uniformly over a disc of radius \(a\) (the density being
\[ d = 1/(\pi a^2) \]. The Hankel transform of the unitary mass distribution function, which is equal to \( d \) inside the disc and null outside it, is given by

\[
D = \frac{1}{2 \pi} \left[ \frac{2 J_1 (k a)}{k a} \right]
\]  
(10)

As \( a \to 0 \), the disc load reduces to a \( \delta \) function. At this limit, \( D = 1/(2 \pi) \). The bracketed quantity in (10) is called the disc factor associated with the finite size load.

Taking a first approximation approach, the surface stress components are:

\[
\tau_{ss} = -g d, \quad \tau_{rz} = 0
\]

and their Hankel transforms are

\[
T_{zz} = \lambda X + 2 \mu \dot{U} = -gD
\]

\[
T_{rz} = \mu (\dot{V} - U k) = 0.
\]

Applying (7) to the above equation, we obtain the boundary condition at the surface

\[
y_2 = -g/2\pi, \quad y_4 = 0.
\]  
(11)

All variables are continuous across the interfaces within the medium.

The medium is assumed to be a uniform half-space beneath some depth \( z_0 \). The solution of equation (9) at the interface \( z_0 \) is taken as the ‘initial solution’ of the problem. The analytical solution for a uniform half-space is given by Farrell (1972)

\[
y_1 = A_1 e^{kz} + (-kz + (\lambda + 3 \mu) / (\lambda + \mu)) B_1 e^{kz}
\]

\[
y_2 = 2\mu k A_1 e^{kz} - 2\mu k (kz - \sigma / (\lambda + \mu)) B_1 e^{kz}
\]

\[
y_3 = -A_1 e^{kz} + kz B_1 e^{kz}
\]

\[
y_4 = -2\mu k A_1 e^{kz} + 2\mu k (kz - \mu / (\lambda + \mu)) B_1 e^{kz}
\]  
(12)

where \( A_1 \) and \( B_1 \) are arbitrary constants. Let \( z = 0 \) in (12). Then the initial solutions are:

\[
\begin{bmatrix}
  k \lambda^* y_1 \\
  y_2 \\
  k \lambda^* y_3 \\
  y_4
\end{bmatrix} =
\begin{bmatrix}
  C_1 \\
  2 \tilde{\mu} (\tilde{C} C_1 - \tilde{C} \tilde{C} 2) / (\lambda + 3 \tilde{\mu}) \\
  C_2 \\
  2 \tilde{\mu} (-\tilde{C} C_1 + \tilde{C} \tilde{C} 2) / (\lambda + 3 \tilde{\mu})
\end{bmatrix}
\]  
(13)

On the right hand size of the equation the parameters pertain to the depth \( z_0 \). \( C_1 \) and \( C_2 \) being arbitrary constants. They are determined by the boundary conditions (11).

4. The transformed viscoelastic equations of motion

From the elastic differential equations given in the second section, the viscoelastic equations are readily derived by use of the well-known correspondence principle. From this principle the Laplace or Fourier time transformed viscoelastic field equations and boundary conditions are formally identical with those for an elastic body of the same
For a Maxwell body its stress-strain equation becomes (Cathles, 1975):

$$\dot{\tau}_{ik} + \mu \eta \left( \tau_{ik} - \tau_n \delta_{ik} / 3 \right) = 2 \mu \dot{e}_{ik} + \lambda e_n \delta_{ik}$$  \hspace{1cm} (14)

where $\eta$ is viscosity coefficient, $e_{ik}$ is the strain tensor which obeys the summation convention, i.e. $e_n = e_{11} + e_{22} + e_{33}$; $\delta_{ik}$ is the unit diagonal tensor; the dot denotes time differential.

Let a tilde denote a Fourier transform variable. The Fourier transform of (14) is:

$$i \omega \tilde{\tau}_{ik} + \mu \tilde{\tau}_{ik} / \eta - \mu \tilde{\tau}_{ik} / (3 \eta) = 2 \mu i \omega \tilde{\varepsilon}_{ik} + \lambda i \omega \tilde{e}_n \delta_{ik} \text{.}$$  \hspace{1cm} (15)

Contracting the tensor (15), we obtain:

$$\tilde{\tau}_{kk} = (3 \lambda + 2 \mu) \tilde{\varepsilon}_{kk} \text{.}$$

Substituting this result back into (15) then given

$$\tilde{\tau}_{ik} = (\lambda + \frac{2}{3} \mu \frac{\mu / \eta}{i \omega + \mu / \eta}) \tilde{\varepsilon}_{ik} + 2 \mu \left( \frac{i \omega}{i \omega + \mu / \eta} \right) \tilde{\varepsilon}_{ik} \text{.}$$

Thus:

$$\tilde{\tau}_{ik} = \tilde{\tau}(\omega) \tilde{e}_n \delta_{ik} + 2 \mu(\omega) \tilde{e}_{ik}$$  \hspace{1cm} (16)

where

$$\tilde{\tau}(\omega) = \frac{i \lambda \mu + \mu k' \eta}{i \omega + \mu \eta} \text{,} \quad k' = \lambda + \frac{2}{3} \mu \text{,}$$  \hspace{1cm} (17)

$$\tilde{\mu}(\omega) = \frac{i \mu \omega}{i \omega + \mu \eta} \text{.}$$

(16) has exactly the same form as the constitutive relation for a Hookean elastic solid, where the Lamé parameters $\lambda$ and $\mu$ become the complex functions $\lambda(\omega)$ and $\mu(\omega)$. They can be separated into a real and an imaginary part

$$\lambda = \lambda^R + i \lambda^I \text{,} \quad \mu = \mu^R + i \mu^I \text{,}$$  \hspace{1cm} (18)

where

$$\lambda^R = \frac{\mu^2 k' + \lambda \omega^2 \eta^2}{\omega^2 \eta^2 + \mu^2} \text{,} \quad \lambda^I = -\frac{2/3 \mu^2 \omega \eta}{\omega^2 \eta^2 + \mu^2} \text{,}$$  \hspace{1cm} (19)

$$\mu^R = \frac{\mu \omega^2 \eta^2}{\omega^2 \eta^2 + \mu^2} \text{,} \quad \mu^I = \frac{\mu^2 \omega \eta}{\omega^2 \eta^2 + \mu^2} \text{.}$$

$\tilde{\gamma}_j (j = 1, ..., 4)$ are also complex numbers related to complex Lamé parameters and

$$\tilde{\gamma}_j = y_j + ix_j \text{,} \quad j = 1, ..., 4 \text{.}$$  \hspace{1cm} (20)

Substituting $\lambda$, $\mu$ and $\tilde{\gamma}_j$ for $\lambda$, $\mu$ and $y_j$ in (8), respectively, then separating $\tilde{\gamma}_j$ into real
and imaginary parts, we have

\[
\frac{\partial}{\partial z} \begin{bmatrix} Y \\ X \end{bmatrix} = \begin{bmatrix} A1 & -A2 \\ A2 & A1 \end{bmatrix} \begin{bmatrix} Y \\ X \end{bmatrix}
\]  

\(21\)

where \(Y = (y1, y2, y3, y4)^T\)
\(X = (x1, x2, x3, x4)^T.\)

Note that the symbol \(Y\) has not the same meaning as in (8).

\[A_1 = \begin{bmatrix}
0 & C_{0/l} & -c_2 k/l & 0 \\
-\omega^2 \varrho & 0 & -\varrho g k & -k \\
k & 0 & 0 & \mu^{R/m} \\
-\varrho g k & c_1 k/l & 4c_3 k^2/l & -\omega^2 \varrho & 0
\end{bmatrix}\]

\[A_2 = \begin{bmatrix}
0 & -d_0/l & -d_1 k/l & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\mu^{l/m} \\
0 & d_1 k/l & 4d_2 k^2/l & 0
\end{bmatrix}\]

where \(c_0 = \lambda^R + 2 \mu^R\).
\(d_0 = \lambda^l + 2 \mu^l.\)

\(c_1 = \lambda^R c_0 + \lambda^l d_0.\)
\(d_1 = \lambda^l c_0 - \lambda^R d_0.\)

\(c_2 = \lambda^l d_0 + \lambda^R c_0.\)
\(d_2 = (\mu^R e + \mu^l f) c_0 - (\mu^R f - \mu^l e) d_0.\)

\(c_3 (\mu^R f - \mu^l e) c_0 + (\mu^R e + \mu^l f) d_0.\)

\(e = \lambda^l + \mu^l.\)
\(f = \lambda^R + \mu^R.\)

\(l = c_0^2 + d_0^2.\)
\(m = (\mu^R)^2 + (\mu^l)^2.\)

Corresponding boundary conditions are:

\(y_2 = -g \varphi/(2\pi)\) \(y_4 = 0.\)

\(x_2 = 0\) \(x_4 = 0\) on the free surface.

\(22\)

All variables are continuous across an interface within the medium, and the initial solutions are:
\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3 \\
  y_4 \\
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
\end{bmatrix} = \begin{bmatrix}
  C_1 \\
  -2\mu^R (C_2 - P^R) + 2\mu^I (C_4 - P^I) \\
  C_2 \\
  2\mu^R (C_2 - P^R) - 2\mu^I (C_4 - P^I) \\
  C_3 \\
  -2\mu^R (C_4 - P^I) - 2\mu^I (C_2 - P^R) \\
  C_4 \\
  2\mu^R (C_4 + P^I) + 2\mu^I (C_2 + P^R) \\
\end{bmatrix}
\]

(23)

where

\[
P^R = \left( \sigma^R \sigma^R + \sigma^I \sigma^I \right) (y_1 + y_3) - (\sigma^I \sigma^R - \sigma^I \sigma^R) (x_1 + x_3) \big/ (\sigma^R)^2 + (\sigma^I)^2
\]

\[
p^I = \left( \sigma^R \sigma^R + \sigma^I \sigma^I \right) (x_1 + x_3) + (\sigma^I \sigma^R - \sigma^I \sigma^R) (y_1 + y_3) \big/ (\sigma^R)^2 + (\sigma^I)^2
\]

\[
\sigma^R = \lambda^R + 2 \mu^R, \quad \sigma^I = \lambda^I + 2 \mu^I,
\]

\[
\sigma^R = \lambda^R + 3 \mu^R, \quad \sigma^I = \lambda^I + 3 \mu^I
\]

\[P^R \text{ and } P^I \text{ are obtained by substituting } \mu^R \text{ and } \mu^I \text{ for } \sigma^R \text{ and } \sigma^I \text{ in } P^R \text{ and } P^I.\]

5. The transformed solutions of the viscoelastic equations

There are two usual methods for solving the differential equations (21); matrix techniques and Runge-Kutta techniques. Here we employ the latter and also make equation (21) non-dimensional for convenience of calculation. From differential equation theory, if we chose four arbitrary constants to be initial solution vectors and relate the corresponding surface solutions to four boundary conditions, we are able to obtain a unique solution. In general, in order to get the solutions, 5 propagations from the depth \( z_o \) to the surface are required. However, because equation (21) is anti-symmetric, the problem can be greatly simplified by the following relation (Li & Hsu 1985):

\[
(Y_3, X_3) = (-X_1, Y_1)
\]

\[
(Y_4, X_4) = (-X_2, Y_2)
\]

where \((Y_i, X_i)\) is the surface solution vector corresponding to zero initial constants, except for \(Ci=1 \ (i=1, 2)\). Letting a linear combination of the four solution vectors satisfy the boundary conditions (22), we find four constants \(C_1, ..., C_4\). The usual method to obtain the solutions is then to propagate the equation (21) from the depth \( z_o \) to the surface once more using the four constants as initial solution vectors. However, it is suggested that the last step can be replaced by a linear combination of the four constants with the four solution vectors for the surface

\[
\begin{bmatrix}
  Y \\
  X \\
\end{bmatrix} = \sum_{i=1}^{4} C_i \begin{bmatrix}
  Y_i \\
  X_i \\
\end{bmatrix}
\]

171
Therefore, the procedure for solving equation (21) is reduced to two time propagations from \( z_0 \) to the surface, and the calculation time is greatly reduced.

It should be stated that the wavenumber \( k \) is in units \( 1/H \), where \( H \) is the thickness of the stratified half-space, in order to avoid the overflow in the computation.

The transformed coefficient solutions have been calculated for two different viscoelastic models. In the first model, the crust and upper mantle elastic parameters of the earth model G-D 1066A of Gilbert and Dziewonski (1975) are used as elastic parameters of the stratified half-space with a thickness of 500 km. Below this there is a homogeneous elastic half-space. The viscosity parameter model is that of Werner (1985) (Fig. 1). In Fig. 2 the real and the imaginary parts of the transformed functions \( kU \) and \( kV \) are drawn as functions of the wavenumber for both \( O \) and \( M_2 \) waves. For graphic convenience in the figure \( \tilde{U} = kU \) and \( \tilde{V} = kV \). Since the real part of the solution is mainly related to the elastic model, there are no differences between the two tidal waves. In order to better visualize the viscosity influence, the same transformed functions have also been drawn in terms of amplitude and phase-lag (Fig. 5). Only small dephasements can be observed for both waves and the phase-lag of the \( O \) wave is large than that of \( M_2 \) wave. The major effects occur in the \( k \) range between 6 and 20, (units of \( k \) in 500 km\(^{-1} \)).

The earth model PREM of Dziewonski and Anderson (1981) is used as second model. Here, the quality factor \( Q \) has been used for the evaluation of the viscosity parameter \( \eta \). \( Q \) is defined by

\[
Q^{-1} = \Delta E(\omega)/2\pi E_0(\omega)
\]

where \( \Delta E(\omega) \) is the energy loss in a cycle at frequency \( \omega \) and \( E_0 \) is the 'elastic' energy stored in the system during the cycle. The energy dissipation \( \Delta E \) is just associated with the imaginary part of the elastic moduli. One can easily find that the quality \( Q_\mu \) due to dissipation in shear may be written as (Kennett, 1983):

\[
Q^{1/3}_\mu = \mu^I/\mu
\]

\( Q_\mu \) is generally less than 0.01. Thus (19) can be rewritten as

\[
\mu^R \to \mu, \quad \mu^I \to \mu Q^{-1} \\
\lambda^R \to \lambda, \quad \lambda^I \to -2\mu^I/3
\]

Here, there is no direct relation between the viscoelastic parameters and the frequencies. The results of the calculations are shown in Figs. 4 and 6. Since the \( Q \) factor is not related to the loading frequency, the results are nearly the same for both the tidal waves. Comparing them with the first model results, the discrepancies between the two models can be found both in the real and in the imaginary parts (or in the amplitudes and in the phases). Discrepancies in the real part are caused by the different assumptions for the elastic models. Discrepancies in the imaginary part are instead due to the different assumptions for the viscosity models.

Having obtained the transformed solutions, Green's functions on the surface can be readily calculated. Figs. 2 and 4 show that if \( k \) becomes large enough \( \tilde{U} \) and \( \tilde{V} \) become constants. In a similar way to Farrell's definition in the spherical case, we define

\[
\lim_{k \to \infty} \begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix} = \begin{bmatrix} U_\infty \\ V_\infty \end{bmatrix}
\]

172
Fig. 1 — The viscosity distribution of the model (unit: Poise), after D. Werner, 1985.

Fig. 2 — The transformed functions for the 1066A elastic model and viscosity model of Fig. 1, and for the $M_2$ and the $O_7$ waves. $R$ denotes real part, $I$ imaginary part. CGS units. $k$ is in units of 500 km $^{-1}$.

Fig. 3 — The quality factor $Q$ distribution of the model PREM.

Fig. 4 — The transformed functions for the model PREM. Units as in Fig. 2.
Fig. 5 — Amplitude and phase of the transformed functions for the model 1066A. A denotes amplitude, P Phase. CGS units, phase in radians.

Fig. 6 — Amplitude and phase of transformed functions for the model PREM. Units as in Fig. 5.

Fig. 7 — Amplitude and phase Green's functions of the displacements for the model PREM. r is distance from the load. CGS units, phase in degree.
Thus the Hankel transform is changed to a limited integral. For the real part of the Green’s functions,

\[
\begin{align*}
\text{displacement} & \quad u^R (r) = \int_0^K (kU - U_\infty) J_0 (kr) \, dk + U_\infty/r \\
\text{gravity} & \quad \Delta g^R (r) = 2g^* \frac{u^R}{a} \\
\text{tilt} & \quad t^R (r) = \int_0^K (kU - U_\infty) kJ_1 (kr) \, dk + U_\infty/r^2 \\
\text{strain} & \quad e^R_{zz} = -\lambda/\sigma \int_0^K (kV - V_\infty) kJ_0 (kr) \, dk + 2\pi\delta (r) V_\infty \\
& \quad e^R_{rr} = -t^R \\
& \quad e^R_{\theta \theta} = \nu^R / r \\
& \quad e^R_{r \theta} = e^R_{\phi \phi} = e^R_{r r} = 0.
\end{align*}
\]

Where \( a \) is the earth’s radius. The magnitude of \( K \) depends on the convergence of the integral. It can be evaluated from the computed values of the real and the imaginary parts of the transformed functions (\( \tilde{U} \) and \( \tilde{V} \)). Beyond a certain \( k \), depending on the model, they become stable, being related to progressively shorter wave lengths and surface layers where no modifications occur in the viscoelastic properties. In our case, for both the rheological models considered, \( K = 500 \) may be a valid limit, judging from Figs. 2 and 4.

In the above computations of Green’s functions (formula 24) the following Bessel integrals are used:

\[
\int_0^\infty J_m (kr) k^{-m+n} \, dk = 2^{-m} r^{m-n-1} \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( m - \frac{n-1}{2} \right)} (-1 < n < 2m+1)
\]

where \( \Gamma \) is the Gamma function. We are able to obtain the imaginary parts of the Green’s functions in same way.

The second terms of the displacement and tilt Green’s functions on the right of (24) are the solutions for a uniform elastic half-space with the parameters of the top layer of the earth model.

The amplitude and phase-lag Green’s functions of the displacement have been calculated by the transformed functions of the model PREM and formula (24). The results in Fig. 7 show that the maximum phase-lag occurs at about 80 km from the load.

The displacements, gravity, tilt and strain components given in (24) have to be Fourier anti-transformed in order to obtain the solution in the original space. Looking at the single \( M_2 \) frequency only, we have
\begin{equation}
\hat{u}_{M_2} = \int_0^{\infty} \bar{u}_{M_2} e^{i\omega t} \, d\omega
\end{equation}

where \( \bar{u}_{M_2} \) and \( u_{M_2} \) are the vertical displacement and its transform respectively for the \( M_2 \) tidal frequency. In the purely elastic case:

\begin{equation}
\hat{u}_{M_2} = \bar{u}_{M_2} e^{i\omega M_2^2}
\end{equation}

Since \( \bar{u}_{M_2} \) is a \( \delta \) function centered at the \( M_2 \) frequency. In the viscoelastic case, due to the dissipation of energy, the \( \delta \) function becomes a bell shaped function, the wideness of the bell depending on the departure from the elastic case. In the present case, related to a lithospheric structure, the departure can be considered negligible and (26) is still valid. For other tidal frequencies the same procedure can be adopted.

6. Conclusion

Two viscoelastic half-space models have been considered in order to evaluate the near-field surface loading effects in the earth tide observations as well as their dephasements caused by departures from the elastic case. A suitable fast computer methodology has been presented for that purpose. The complex Hankel transforms of the vertical and horizontal surface displacements are given as wavenumber functions in Figs. 2, 4, 5, and 6 for the two adopted models and for the \( O_1 \) and \( M_2 \) tidal frequencies. The two models adopted are different in both their elastic and viscous characteristics. By considering the transformed functions of the vertical and horizontal displacements (Fig. 5), no significant amplitude differences occur between the \( M_2 \) and the \( O_1 \) wave, which are mostly tied to the elastic properties of the models. At large \( k \) wavenumbers they give the solutions of the elastic half-space with elastic parameters of the upper crust. Since the two elastic models are different in the crust, both the vertical and the horizontal responses for the first model at large \( k \) values are almost twice those obtained for the second model. The dephasements caused by the viscosity models also appear very different. For the second model, no phase differences occur between the diurnal and semidiurnal waves. This is due to the fact that the \( Q \) model is frequency independent. In both cases, the maximum transformed phase lag occurs at about \( k = 10 \), and is mainly affected by the low viscous asthenosphere. A better insight is given by the phase lag Green’s functions (Fig. 7). From Fig. 7 we find that for the model PREM the maximum phase-lag occurs at about 80 km from the load, this distance depending on the depth of the assumed low viscous asthenosphere. This result is similar to the result Zschau (1978) obtained by using a viscoelastic spherical earth model. Zschau’s result corresponds to near ocean ridges and near subduction zones, thus the loading tide phase-lag obtained by Zschau is about one order of magnitude greater than ours.

Acknowledgments. Thanks are due to one of the unknown referees for his valuable suggestions and to C.D. Macbeth for reading the first English version of the manuscript. The first author (W. Mao) has carried out this work with the partial support of the “ICTP Programme for Training and Research in Italian Laboratories, Trieste, Italy” in his doctoral training at The Institute of Geodesy and Geophysics of Trieste University. This work has been supported by MPI 40% C. 85, 6HO. 2/12/2 Prof. M. Zadro funds.

This paper was presented at the workshop “Geodynamics and Tidal Measurements”, EGS and ESC Joint Meeting, Kiel, 1986.
References


Boussinesq J.; 1885: Application des potentiels a l’étude de l’équilibre et du mouvement des solides elasti-


Li G. and Hsu H.; 1985: Response of visco-elastic layered spherical earth under the surface load. 10th Symp. on Earth Tides.


Zschau J.; 1977: Phase shift of tidal sea load deformation of the earth’s surface due to low viscosity layers in the interior. 8th Symp. on Earth Tides.